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The analytic solutions of some boundary layer problems in the theory of Brownian motion

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Abstract. We give analytic solutions of the generalised albedo and Milne problems for the Uhlenbeck-Ornstein process, and we discuss the relation between these and the original Wang-Uhlenbeck problem of determining the distribution of first-passage times.

1. Introduction

In 1945 Wang and Uhlenbeck [1] posed the problem of finding the distribution of position, x, and velocity, u, for one-dimensional Brownian motion in a uniform force field starting at $x=y$ and $u=v$ in the presence of an absorbing barrier at $x=0$. It can be posed as a boundary value problem for $W(x, u; t|y, v)$ satisfying

$$\frac{\partial W}{\partial t}=\frac{\partial^2 W}{\partial u^2}+(u+2\alpha)\frac{\partial W}{\partial u}+W-u\frac{\partial W}{\partial x} \qquad x>0 \tag{1.1}$$

$$W(x, u; 0)=\delta(x-y)\delta(u-v) \tag{1.2}$$

$$W\to 0 \qquad \text{as} \qquad u\to\pm\infty \qquad \text{and as} \qquad x\to+\infty \tag{1.3}$$

$$W(0, u; t)=0 \qquad u>0. \tag{1.4}$$

The quantities t, u, x and α have been made dimensionless as in [2].

This problem is still not completely solved, but recently [2] we showed how to obtain the Laplace transform $\bar{W}(x, u; p)$. More precisely, we obtained the quantity

$$\bar{\Phi}(y, v; p)=-\int_{-\infty}^{\infty} u\bar{W}(0, u; p|y, v)\, \mathrm{d}u \tag{1.5}$$

from which it is possible to deduce the mean of the first-passage time distribution, and also the asymptotic distribution for large t. Here we obtain explicitly the expression for $\bar{W}$ and relate it to the analytic solutions of two stationary problems [3, 4] which have arisen in chemical kinetics and coagulation studies. We also present a simplified calculation of the mean first-passage time.

The stationary problems involve finding $P(x, u)$ satisfying

$$\frac{\partial^2 P}{\partial u^2}+(u+2\alpha)\frac{\partial P}{\partial u}+P-u\frac{\partial P}{\partial x}=0 \qquad x>0 \tag{1.6}$$

$$P\to 0 \qquad \text{as} \qquad u\to\pm\infty \tag{1.7}$$

$$P(0, u)=g(u) \qquad u>0. \tag{1.8}$$

We obtain a problem with a unique solution in one of two ways. The first way may be called the generalised albedo problem (GAP) and consists of adding the condition

$$P \to 0 \qquad \text{as} \qquad x \to +\infty. \tag{1.9}$$

The second way is called the Milne problem (MP). It concerns the zero-field case ($\alpha = 0$) and consists of taking $g(u) = 0$, with a certain specified asymptotic behaviour for large x:

$$P(x, u) \sim (2\pi)^{-1/2}(x - u + M)\exp(-\tfrac{1}{2}u^2) \qquad \text{as } x \to +\infty \qquad g(u) = 0 \tag{1.10}$$

where M is a constant, called the Milne length, which is to be determined. There is a connection [3] between these two problems. We denote by $a(x, u|v)$ the solution of GAP with $g(u) = \delta(u - v)$. Then the solution of the Milne problem is

$$P^M(x, y) = (2\pi)^{-1/2}(x - u)\exp(-\tfrac{1}{2}u^2) + (2\pi)^{-1/2}\int_0^\infty v \exp(-\tfrac{1}{2}v^2)a(x, u|v)\, dv. \tag{1.11}$$

Until our recent analytical result, most of the progress, even on the equilibrium problems [3-6], was numerical; for example, the Milne length was found to lie in the region of 1.46, but this quantity was not recognised as $-\zeta(\tfrac{1}{2}) = 1.460\,35\ldots$. In this paper we obtain analytic expressions for the solutions of both GAP and MP. This enables us to assess the accuracy of the numerical solutions, in particular near the singular point $(x, u) = (0, 0)$.

The case $\alpha \neq 0$ is, in many ways, easier than the case $\alpha = 0$, so we shall solve the GAP for $\alpha \neq 0$ and obtain the zero-field result by considering the limit $\alpha \to 0$. The appropriate set of eigenfunctions is

$$\{f_n^\pm(u) \colon n = 0, 1, 2 \ldots\} \tag{1.12}$$

where

$$f_n^+(u) = f_n(u) \qquad f_n^-(u) = f_n(-u) \tag{1.13}$$

and

$$\begin{aligned} f_n(u) &= (n!)^{-1/2}\exp(\tfrac{1}{4}u^2)D_n(2q_n - u) \\ &= (n!)^{-1/2}\exp[\tfrac{1}{2}(q_n - u)^2 + \tfrac{1}{2}q_n^2]\frac{d^n}{du^n}\exp[-\tfrac{1}{2}(2q_n - u)^2] \end{aligned} \tag{1.14}$$

where

$$q_n = (n - \alpha^2)^{1/2} \tag{1.15}$$

and $D_n(z)$ is the parabolic cylinder function, properties of which are given in appendix A of [2].

We shall assume, as has become traditional, that

(a) the set $\{f_n^\pm\}$ is complete over the range $(-\infty, \infty)$ with scalar product

$$(F, G) = \int_{-\infty}^{\infty} |u| \exp(-\tfrac{1}{2}u^2)F(u)G(u)\, du \tag{1.16}$$

(b) the set $\{f_n^+\}$ is complete over the range $(0, \infty)$ with the scalar product

$$(F, G)^+ = \int_0^\infty u \exp(-\tfrac{1}{2}u^2)F(u)G(u)\, du. \tag{1.17}$$

The corresponding results for $\alpha = 0$ have been proved [7] and we think it is only a matter of time for someone to extend the proof to our case. We have ourselves established [2] that, for all n, f_n^- has an expansion in terms of $\{f_n^+\}$ for $u > 0$:

$$f_n^-(u) = \sum_{m=0}^{\infty} \tfrac{1}{2} q_m^{-1} \sigma_{mn} f_m^+(u) \qquad u > 0 \tag{1.18}$$

which means that (b) follows from (a). A direct proof of (1.18) is given in the appendix.

2. The generalised albedo solution

It is convenient to introduce the pseudoproduct

$$[F, G] = \int_{-\infty}^{\infty} u \exp(-\tfrac{1}{2}u^2) F(u) G(u) \, du. \tag{2.1}$$

The elements of the set (1.12) satisfy

$$\begin{aligned} [f_m^+, f_n^+] &= (8\pi)^{1/2} q_n \delta_{mn} \\ [f_m^-, f_n^-] &= -(8\pi)^{1/2} q_n \delta_{mn} \\ [f_m^+, f_n^-] &= 0. \end{aligned} \tag{2.2}$$

Then, given our completeness assumption, an arbitrary function $G(u)$ may be expanded as

$$G(u) = \sum_{n=0}^{\infty} [G_n^+ f_n^+(u) + G_n^- f_n^-(u)] \tag{2.3}$$

where

$$G_n^{\pm} = \pm (8\pi)^{1/2} q_n^{-1} [f_n^{\pm}, G] \tag{2.4}$$

and in particular

$$\delta(u - v) = (8\pi)^{-1/2} v \exp(-\tfrac{1}{2}v^2) \sum_{n=0}^{\infty} q_n^{-1} [f_n^+(u) f_n^+(v) - f_n^-(u) f_n^-(v)]. \tag{2.5}$$

Using (1.18) we deduce that

$$\delta(u - v) = (8\pi)^{-1/2} v \exp(-\tfrac{1}{2}v^2) \sum_{n=0}^{\infty} q_n^{-1} f_n^+(u) F_n^+(v) \qquad u > 0 \tag{2.6}$$

where

$$F_n^+(v) = f_n^+(v) - \sum_{m=0}^{\infty} \tfrac{1}{2} q_m^{-1} \sigma_{nm} f_m^-(v). \tag{2.7}$$

Now a set of solutions of (1.6) satisfying (1.7) and (1.9) may be obtained by separation of the variables:

$$P_n(x, u) = \exp(-\tfrac{1}{2}u^2 - \alpha x - \alpha u) f_n^+(u) \exp(-q_n x). \tag{2.8}$$

Hence the solution of the generalised albedo problem is

$$a(x, u|v) = (8\pi)^{-1/2} v \exp[-\tfrac{1}{2}u^2 - \alpha x + \alpha(v - u)] \sum_{n=0}^{\infty} q_n^{-1} f_n^+(u) F_n^+(v) \exp(-q_n x). \tag{2.9}$$

The coefficients appearing in (2.7) take the values [2]

$$\sigma_{mn}=(q_m+q_n)^{-1}Q_m^{-1}Q_n^{-1} \tag{2.10}$$

where

$$Q_n=\lim_{N\to\infty}[n!(N-1)!]^{1/2}\exp(2N^{1/2}q_n)\left(\prod_{r=0}^{N+n-1}(q_r+q_n)\right)^{-1}. \tag{2.11}$$

Since $\sigma_{mn}=\sigma_{nm}$ it follows, from (1.18), that

$$F_n^+(v)=0 \qquad v<0. \tag{2.12}$$

We obtained asymptotic expansions for Q_n and $f_n^\pm$ when $n\to\infty$, while for small n we obtained an asymptotic expression for the denominator of (2.11) as $N\to\infty$. We have therefore succeeded in overcoming the computational difficulties associated with the extremely slow convergence of (2.9).

To obtain the zero-field solution, $\alpha\to 0$, we must pay special attention to the first term of the summation in both (2.7) and (2.9), since $q_0=|\alpha|$ appears in the denominator. From the definition of $f_n^\pm$, it follows that

$$f_0^\pm(v)=1\pm|\alpha|v+O(\alpha^2) \tag{2.13}$$

and the behaviour of Q_0 for small α is given [2] by

$$2|\alpha|Q_0=1-|\alpha|\zeta(\tfrac{1}{2})+O(\alpha^2). \tag{2.14}$$

Hence

$$F_n^+(v)=f_n^+(v)-\sum_{m=1}^{\infty}\frac{f_m^-(v)}{2(m+m^{1/2}n^{1/2})Q_mQ_n}-\frac{1}{n^{1/2}Q_n}+O(\alpha) \tag{2.15}$$

$$F_0^+(v)=2|\alpha|\left(v-\zeta(\tfrac{1}{2})-\sum_{m=1}^{\infty}\frac{f_m^-(v)}{2mQ_m}\right)+O(\alpha^2) \tag{2.16}$$

where now $q_n=n^{1/2}$, so the zero-field limit of (2.9) is

$$a(x,u|v)=\frac{v\exp(-\frac{1}{2}u^2)}{(2\pi)^{1/2}}\left[v-\zeta(\tfrac{1}{2})-\sum_{n=1}^{\infty}\frac{f_n^-(v)}{2nQ_n}+\sum_{n=1}^{\infty}f_n^+(u)\exp(-xn^{1/2})\right.$$
$$\left.\times\left(\frac{f_n^+(v)}{2n^{1/2}}-\frac{1}{2nQ_n}-\sum_{m=1}^{\infty}\frac{f_m^-(v)}{4(mn^{1/2}+m^{1/2}n)Q_mQ_n}\right)\right]. \tag{2.17}$$

The connection between our generalised albedo function and Selinger and Titulaer's [3] function $A(u|v)$ is

$$A(u|v)=ua(0,-u|v)v^{-1} \tag{2.18}$$

which has the property that

$$A(u|v)=0 \qquad u<0. \tag{2.19}$$

When integrated with respect to u, this gives the quantity

$$\phi(v)=1-\int_{-\infty}^{\infty}A(u|v)\,du \tag{2.20}$$

which is the probability that a particle, injected with velocity v at $x=0$, is absorbed at some later time. Substituting (2.9) in (2.18) gives

$$A(u|v)=(8\pi)^{-1/2}u\exp(-\tfrac{1}{2}u^2+\alpha v+\alpha^2)f_0^\pm(u)\sum_{n=0}^{\infty}q_n^{-1}f_n^+(u)F_n^+(v) \tag{2.21}$$

where the $+(-)$ sign is taken if α is negative (positive). Then, using the pseudo-orthogonality relations (2.2),

$$\phi(v)=\begin{cases}1 & \alpha>0\\ 1-\exp(\alpha v+\alpha^2)F_0^+(v) & \alpha<0\end{cases} \tag{2.22}$$

and (2.16) shows that this quantity is continuous, though not differentiable, at $\alpha=0$. We therefore verify that the Uhlenbeck-Ornstein process is recurrent for $\alpha>0$ and non-recurrent for $\alpha<0$.

3. The Milne solution

We must now substitute (2.17) in (1.11). The integration is facilitated by property (2.12), which allows us to extend the range of integration to $-\infty$. Now, for $\alpha\neq 0$, the pseudo-orthogonality relations (2.2) give

$$[\tfrac{1}{2}(f_0^+ + f_0^-), f_n^\mp]=0$$
$$[\tfrac{1}{2}\alpha^{-1}(f_0^+ - f_0^-), f_n^\mp]=0 \qquad n=1,2,3,\ldots. \tag{3.1}$$

Hence, in the limit $\alpha\to 0$, we obtain

$$\int_{-\infty}^{\infty} v\exp(-\tfrac{1}{2}v^2)f_n^\mp(v)\,\mathrm{d}v=0$$

and

$$\int_{-\infty}^{\infty} v^2\exp(-\tfrac{1}{2}v^2)f_n^\mp(v)\,\mathrm{d}v=0 \qquad \alpha=0. \tag{3.2}$$

We therefore find that

$$P^M(x,u)=(2\pi)^{-1/2}\exp(-\tfrac{1}{2}u^2)\times\left(x-u-\zeta(\tfrac{1}{2})-\tfrac{1}{2}\sum_{n=1}^{\infty} n^{-1}Q_n^{-1}f_n^+(u)\exp(-xn^{1/2})\right). \tag{3.3}$$

This is the solution of the Milne problem, and, as stated in the introduction, it establishes that the Milne length is equal to $-\zeta(\frac{1}{2})$.

Selinger and Titulaer [3] and Titulaer [4] studied the density profile

$$n^M(x)=\int_{-\infty}^{\infty} P^M(x,u)\,\mathrm{d}u = x-\zeta(\tfrac{1}{2})-\sum_{n=1}^{\infty} a_n\exp(-xn^{1/2}) \tag{3.4}$$

where

$$a_n=\tfrac{1}{2}(n!)^{-1/2}n^{1/2n-1}\exp(-\tfrac{1}{2}n)Q_n^{-1}. \tag{3.5}$$

For Q_n, as $n\to\infty$, we have [2] the asymptotic expansion

$$Q_n\sim(2\pi)^{1/2}\exp\left(-\sum_{m=0}^{\infty}\frac{\zeta(-m-\frac{1}{2})}{(2m+1)n^{m+1/2}}\right). \tag{3.6}$$

Combining this with the Stirling expansion for $n!$, we obtain

$$a_n \sim \tfrac{1}{2}(2\pi)^{-3/4} n^{-5/4} \exp\left(\sum_{m=0}^{\infty} \frac{\zeta(-m-\frac{1}{2})}{(2m+1)n^{m+1/2}} - \frac{1}{2} \sum_{r=1}^{\infty} \frac{(-1)^{r-1} Br}{2r(2r-1)n^{2r-1}} \right) \tag{3.7}$$

$$\sim \sum_{k=1}^{\infty} c_k n^{-k/2-3/4}. \tag{3.8}$$

The first few coefficients in this expansion are

$$c_1 = 0.125\,989\,72 \qquad c_2 = -0.026\,191\,53 \qquad c_3 = -0.002\,527\,14$$
$$c_4 = -1.6763 \times 10^{-4} \qquad c_5 = 2.2823 \times 10^{-4}. \tag{3.9}$$

For small n we put

$$a_n = \sum_{k=1}^{5} c_k n^{-k/2-3/4} + b_n \tag{3.10}$$

where the first few b_n are

$$b_1 = 3.0447 \times 10^{-4} \qquad b_2 = 2.273 \times 10^{-5} \qquad b_3 = 4.83 \times 10^{-6}$$
$$b_4 = 1.60 \times 10^{-6} \qquad b_5 = 6.7 \times 10^{-7}. \tag{3.11}$$

We then have

$$n^M(x) = x - \zeta(\tfrac{1}{2}) - \sum_{k=1}^{5} c_k Z(\tfrac{1}{2}k + \tfrac{3}{4}, x) - \sum_{n=1}^{\infty} b_n \, e^{-x\sqrt{n}} \tag{3.12}$$

where

$$Z(\beta, x) = \sum_{n=1}^{\infty} n^{-\beta} \, e^{-x\sqrt{n}} \tag{3.13}$$

is the function defined by Titulaer which, for small x, using the method of Olver [8], may be computed as

$$Z(\beta, x) = 2\Gamma(2-2\beta)x^{2\beta-2} + \sum_{n=0}^{\infty} \zeta(\beta - \tfrac{1}{2}n) \frac{(-x)^n}{n!}. \tag{3.14}$$

An efficient programme for $n^M(x)$ is now obtained by using (3.13) for $x \geq 1$ and (3.14) for $x < 1$ and substituting in (3.12). Titulaer calls $x - \zeta(\frac{1}{2}) - n^M(x)$ the 'boundary layer part of the density profile' and in table 1 we give our values for this quantity with Titulaer's values printed alongside for comparison.

Table 1. The boundary layer part of the density profile $x - \frac{1}{2} - n^M(x)$: (A), our values; (B), Titulaer's values.

x	(A)	(B)
0	0.524 24	0.528
2^{-8}	0.470 56	0.476
2^{-6}	0.415 16	0.427
2^{-4}	0.333 86	0.341
2^{-2}	0.201 20	0.206
1	0.060 11	0.061
4	0.001 98	0.002

The general outline of the density profile is as obtained by Titulaer. Indeed the leading terms of his approximation are given by his equation (4.4):

$$n^M(x) = x + x^M - 0.1216 Z(\tfrac{5}{4}, x) \qquad \text{Titulaer}$$

where we have corrected an obvious error of sign. Our solution has this same form, but the Milne length is now identified as $-\zeta(\frac{1}{2})$ and the next term has $c_1 = \frac{1}{2}(2\pi)^{-3/4} = 0.125\,98\ldots$ in place of 0.1216.

We can also use the method of the appendix to [9] to show that as $u \to 0^+$

$$P^M(0, -u) = a\{u^{1/2} - \tfrac{9}{20}u^{5/2} + O(u^{7/2})\} \tag{3.15}$$

where

$$a = 2\sqrt{3}(2\pi)^{-3/4} = 0.8729$$

a result that was given by Titulaer [4] with a estimated as 0.8425 ± 0.0325.

4. The Green function

The Laplace transform of the original Wang–Uhlenbeck problem is

$$\frac{\partial^2 \bar{W}}{\partial u^2} + (u + 2\alpha)\frac{\partial \bar{W}}{\partial u} + (1-p)\bar{W} - u\frac{\partial \bar{W}}{\partial x} = -\delta(x-y)\delta(u-v) \qquad x > 0 \tag{4.1}$$

$$\bar{W} \to 0 \qquad \text{as} \qquad u \to \pm\infty \qquad \text{and as} \qquad x \to +\infty \tag{4.2}$$

$$\bar{W}(0, u; p) = 0 \qquad u > 0. \tag{4.3}$$

If (4.3) is replaced by

$$\bar{W} \to 0 \qquad \text{as} \qquad x \to -\infty \tag{4.3a}$$

the solution is

$$\bar{W}^F(x, u; p) = (8\pi)^{-1/2} \exp[-\tfrac{1}{2}u^2 + \alpha(v-u) + \alpha(y-x)] \times \sum_{n=0}^{\infty} q_n^{-1} \exp(-q_n|x-y|) \begin{cases} f_n^-(u) f_n^-(v) & x < y \\ f_n^+(u) f_n^+(v) & x > y \end{cases} \tag{4.4}$$

where the functions $f_n^\pm$ are as defined in § 1, except that

$$q_n = (n + \alpha^2 + p)^{1/2}. \tag{4.5}$$

With this change in q_n, the functions $P_n(x, u)$ defined by (2.8) are solutions of the homogeneous equation of (4.1) satisfying (4.2). Hence, following the same procedure as in § 2, we obtain the solution of (4.1) to (4.3) by adding a suitable linear combination of $P_n(x, u)$ to $\bar{W}^F$:

$$\bar{W}(x, u; p) = \bar{W}^F(x, u; p) - (32\pi)^{-1/2} \exp[-\tfrac{1}{2}u^2 + \alpha(v-u) + \alpha(y-x)] \times \sum_{m,n=0}^{\infty}\sum q_m^{-1} q_n^{-1} \sigma_{mn} \exp(-q_m x - q_n y) f_m^+(u) f_n^-(v). \tag{4.6}$$

For $x = 0$ this quantity reduces to

$$\bar{W}(0, u; p|y, v) = (8\pi)^{-1/2} \exp[-\tfrac{1}{2}u^2 + \alpha(v-u) + \alpha y] \sum_{n=0}^{\infty} q_n^{-1} \exp(-q_n y) F_n^-(u) f_n^-(v) \tag{4.7}$$

and, substituted into (1.5), it gives the Laplace transform of the first-passage-time distribution, which is the quantity we calculated in our previous paper [2]. On the other hand, for $y=0+$ it becomes

$$\bar{W}(x, u; p|0+, v)=(8\pi)^{-1/2}\exp[-\tfrac{1}{2}u^2+\alpha(v-u)-\alpha x]\sum_{n=0}^{\infty} q_n^{-1}\exp(-q_n x)f_n^+(u)F_n^+(v) \tag{4.8}$$

which, for $p=0$, is v^{-1} times the solution (2.9) of the generalised albedo problem.

In [2], however, we showed that there is a simpler route to the first-passage-time problem than by calculating the Green function. If we study the backward, instead of the forward, Fokker–Planck equation, the problem may be formulated without any reference to the dependence on u and x. We could say that this problem is an essentially backward one, just as the Milne problem is an essentially forward one.

The mathematical complexity of [2] is due mainly to the procedure we used in obtaining the expansion coefficients σ_{mn}. The result is given here in (2.10). Armed with this, and using the completeness assumptions of § 2, we can now present a greatly simplified argument. The mean first-passage time is defined, for $\alpha>0$ only, by

$$\Psi(y, v)=-\bar{\Phi}_p(y, v; 0+). \tag{4.9}$$

This satisfies the differential equation

$$\frac{\partial^2\Psi}{\partial v^2}-(v+2\alpha)\frac{\partial\Psi}{\partial v}+v\frac{\partial\Psi}{\partial y}=-1 \tag{4.10}$$

with the boundary condition

$$\Psi(0, v)=0 \qquad v<0 \tag{4.11}$$

and certain appropriate boundary conditions as $|v|$ and y tend to $+\infty$ (see [2] for details of these). Now put

$$\Psi(y, v)=\tfrac{1}{2}\alpha^{-1}(y+v)+\psi(y, v). \tag{4.12}$$

Then

$$\frac{\partial^2\psi}{\partial v^2}-(v+2\alpha)\frac{\partial\psi}{\partial v}+v\frac{\partial\psi}{\partial y}=0 \tag{4.13}$$

and

$$\psi(0, v)=\tfrac{1}{2}\alpha^{-1}v \qquad v<0. \tag{4.14}$$

The appropriate eigenfunctions are

$$\psi_n(y, v)=e^{\alpha v}f_n^-(v)\exp[(\alpha-q_n)y] \qquad n=0, 1, 2, \ldots \tag{4.15}$$

and the solution is

$$\psi=\sum_{n=0}^{\infty} c_n\psi_n \tag{4.16}$$

where

$$-\tfrac{1}{2}\alpha^{-1}v\,e^{-\alpha v}=\sum_{n=0}^{\infty} c_n f_n^-(v) \qquad v<0. \tag{4.17}$$

Now the expansion coefficients for the left-hand side of (4.17) may be obtained with the pseudo-orthogonality relations (2.2):

$$-\tfrac{1}{2}\alpha^{-1}v\,\mathrm{e}^{-\alpha v}=\mathrm{e}^{\alpha^2}(1+\tfrac{1}{2}\alpha^{-2})f_0^-(v)-\tfrac{1}{2}b_0\alpha^{-1}f_0^+(v)-\tfrac{1}{2}\sum_{n=1}^{\infty}q_n^{-1}(a_nf_n^-(v)+b_nf_n^+(v)) \quad (4.18)$$

where

$$a_n=(n!)^{-1/2}\exp(-\tfrac{1}{2}n+\alpha q_n)(q_n-\alpha)^{n-1} \quad (4.19)$$

$$b_n=(n!)^{-1/2}\exp(-\tfrac{1}{2}n-\alpha q_n)(q_n+\alpha)^{n-1}. \quad (4.20)$$

The expansion (4.18) holds for *all* v. But for negative v we can use (1.18) to express f_n^+ as a sum over f_n^-. We then find that the coefficients in (4.17) are

$$c_0=\mathrm{e}^{\alpha^2}(1+\tfrac{1}{4}\alpha^{-2})-\tfrac{1}{4}\alpha^{-1}\sum_{n=0}^{\infty}q_n^{-1}\sigma_{0n}b_n \quad (4.21)$$

$$c_n=-\tfrac{1}{2}q_n^{-1}\left(a_n+\tfrac{1}{2}\sum_{m=0}^{\infty}q_m^{-1}\sigma_{mn}b_n\right) \quad (4.22)$$

which gives us the same result as (4.4) in [2].

Appendix. Proof of equation (1.18)

In [2] the function

$$\gamma_+(\mathrm{i}q;\tau)=(\Gamma(\tau))^{1/2}\exp[-\tfrac{1}{2}\psi(\tau)q^2-\zeta(\tfrac{1}{2},\tau)q]\prod_{m=0}^{\infty}\left(\frac{\exp(q/q_m-q^2/2q_m^2)}{1+q/q_m}\right) \quad (\mathrm{A1})$$

was studied, where $q_m=(m+\tau)^{1/2}$ and the τ plane is cut along the negative real axis. This function is regular in q, except for poles at $q=-q_m$, and satisfies

$$\gamma_+(\mathrm{i}q;\tau)\gamma_+(-\mathrm{i}q;\tau)=\Gamma(\tau-q^2). \quad (\mathrm{A2})$$

It was shown in [2] that as $|q|\to\infty$, with $|\arg q|\leqslant\pi-\delta$,

$$\gamma_+(\mathrm{i}q;\tau)\sim(2\pi)^{1/4}q^{-q^2+\tau-1/2}\,\mathrm{e}^{q^2/2} \quad (\mathrm{A3})$$

and also that the parabolic cylinder function

$$D_{q^2-\tau}(2q+u)\sim(2\pi)^{1/2}q^{q^2-\tau+1/3}\,\mathrm{e}^{-q^2/2}\mathrm{Ai}(q^{1/3}u) \quad (\mathrm{A4})$$

where Ai denotes the Airy function.

It follows that

$$f(q;\tau,u)=\gamma_+(\mathrm{i}q;\tau)D_{q^2-\tau}(2q+u) \quad (\mathrm{A5})$$

is regular except for poles at $q=-q_m$, and that

$$f(q;\tau,u)\sim(2\pi)^{3/4}q^{-1/6}\mathrm{Ai}(q^{1/3}u) \quad (\mathrm{A6})$$

as $|q|\to\infty$ with $|\arg q|\leqslant\pi-\delta$. Thus $f(q)$ is exponentially small at infinity when $u>0$, and is $\mathrm{O}(|q|^{-1/6})$ when $u=0$, except when q is near the negative real axis.

The behaviour of $D_{q^2-\tau}(-2q+u)$ was also obtained in [2]. With the aid of (A2) and the asymptotic form of the Airy function it is found that

$$f(-q;\tau,u) \sim \left(\frac{\pi}{2qu}\right)^{1/4} \frac{\cos[(q^2-\tau+\frac{1}{3})\pi+\frac{2}{3}q^{1/2}u^{3/2}]}{\sin(\tau-q^2)\pi} \qquad \text{if } u>0$$
$$f(-q;\tau,0) \sim (2\pi)^{3/4}\mathrm{Ai}(0)q^{-1/6}\frac{\cos(q^2-\tau+\frac{1}{3})\pi}{\sin(\tau-q^2)\pi} \tag{A7}$$

as $|q|\to\infty$ with $|\arg q|\leq\frac{1}{2}\pi-\delta$.

Now consider

$$I_N=\frac{1}{2\pi \mathrm{i}}\int_{C_N}\frac{f(q;\tau,u)}{q-(n-\tau)^{1/2}}\,\mathrm{d}q \tag{A8}$$

where C_N is the circle $|q|=|\tau+N+\frac{1}{2}|^{1/2}$ and N is a large positive integer. On C_N, $|\sin(\tau-q^2)\pi|\geq C$ for some $C>0$, so that the integrand is $O(N^{-5/8})$ if $u>0$ and $O(N^{-7/12})$ if $u=0$. Consequently $I_N\to 0$ as $N\to\infty$. When we evaluate I_N as a sum of residues with the aid of (A2), we obtain

$$\gamma_+(\mathrm{i}q_n;\tau)D_n(2q_n+u)=\sum_{m=0}^{\infty}\frac{D_m(2q_m-u)}{2q_m(q_m+q_n)m!\,\gamma_+(\mathrm{i}q_m;\tau)} \tag{A9}$$

where the remainder of the series after N terms is $O(N^{-1/8})$ if $u>0$, and $O(N^{-1/12})$ if $u=0$. When n is a positive integer and $\tau=\alpha^2$ ($\tau=p+\alpha^2$ for § 4), (A9) gives (1.18); however, both sides of (A9) are analytic in q_n, so that it remains valid for complex n and τ, provided q_n is bounded away from $-q_m(m=0,1,2,\ldots)$. The restriction to $u\geq 0$ is essential, since the series diverges when $0<|\arg u|<\frac{2}{3}\pi$.

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